Project topics

Class (not) on July 4

Last time: - ket-bra notation

- orthogonal projections

- Quantum computer

Problem summary

$\Pi = \sum_{k=1}^{r} |v_k\rangle\langle v_k|$, so in particular

if we use an orthonormal basis for V,

then we get $\mathbb{1} = Id_V = \sum_{k=1}^{n} |e_k\rangle\langle e_k|$

Quantum computers

- If we apply a unitary $\mathcal{U}$ to a system with current state ℓ, then the current state becomes $\mathcal{U}\ell$

- If we ask whether the current state ℓ is contained in a subspace $W \subseteq V$, then there are two options, from which the system will choose randomly.

Pick a statevector $|\psi\rangle$ for ℓ

Let Π be the orthogonal projection onto W

- With probability equal to
$$\frac{\langle \Pi\psi | \Pi\psi\rangle}{\langle \psi | \psi\rangle},$$
the computer will output yes and then change the current state to $\operatorname{span}\{\Pi|\psi\rangle\} = \Pi\ell$

- With probability equal to
$$\frac{\langle \Pi^{\perp}\psi | \Pi^{\perp}\psi\rangle}{\langle \psi | \psi\rangle},$$
the computer will output no and then change the current state to $\operatorname{span}\{\Pi^{\perp}|\psi\rangle\} = \Pi^{\perp}\ell$

Problem summary

If we are in $\mathbb{C}^3$ and our initial state is $|e_0\rangle + 2|e_1\rangle + |e_2\rangle$ and we ask

Is the current state contained in $\mathbb{C}|e_0\rangle$?

$\Pi = |e_0\rangle\langle e_0|$

$|\psi\rangle = |e_0\rangle + 2|e_1\rangle + |e_2\rangle$

Compute probabilities:

$$\langle\psi|\psi\rangle = (\langle e_0| + 2\langle e_1| + \langle e_2|) \cdot (|e_0\rangle + 2|e_1\rangle + |e_2\rangle)$$

$$= 1^2 + 2^2 + 1^2 = 6$$

$$\Pi|\psi\rangle = |e_0\rangle\langle e_0| \, (|e_0\rangle + \ldots)$$

$$= |e_0\rangle$$

$$\langle\Pi\psi|\Pi\psi\rangle = 1$$

→ Probability of yes is $\frac{1}{6}$.

In this case, the current state becomes $\mathbb{C}|e_0\rangle$

If we then ask whether we are in $\mathbb{C}|e_1\rangle$ or in $\mathbb{C}|e_2\rangle$ then we will 100% get no

Thm: Let $V = V_1 \oplus \ldots \oplus V_r$ be an orthogonal decomposition. Perform a sequence of measurements checking whether the current state is in V_k for each $k = 1 \ldots r$. Then the answer yes will occur exactly one time in the sequence. Let p_k be the probability that the unique yes occurs for V_k. Then $p_1 + \ldots + p_r = 1$, and also the list $(p_1, \ldots, p_r)$ is independent of the order we perform our measurements.

As a result, we can introduce a new form of measurement.

- Given an orthogonal decomposition

$$V = V_1 \oplus \cdots \oplus V_r,$$

we can ask

Which of these subspaces contains the current state?

- The answer will be V_k with probability

$$\frac{\langle \Pi_k \psi | \Pi_k \psi \rangle}{\langle \psi | \psi \rangle},$$

where Π_k is the orth. proj. onto V_k and $|\psi\rangle$ is any state vector for the current state.

If the answer is V_k then the current state becomes $\mathbb{C}\,\Pi_k|\psi\rangle = \Pi_k \ell$

This is the measurement relative to $V = V_1 \oplus \dots \oplus V_r$

Similarly, given an orthonormal basis

$$e_1, \dots, e_n$$

of V, the measurement relative to that basis is the one relative to $V = \mathbb{C}e_1 \oplus \dots \oplus \mathbb{C}e_n$.

Corresponding question:

Which of $e_1, \dots, e_n$ spans the current state?

Next class we will learn how to induce an orthogonal decomposition from "nice" operators, such as unitary operators, using eigenspaces.

Quantum Circuits

Let's learn how to visually depict quantum programs.

Let $V = \mathbb{C}^{2^n}$, the n-qubit quantum computer, with basis indexed in binary from

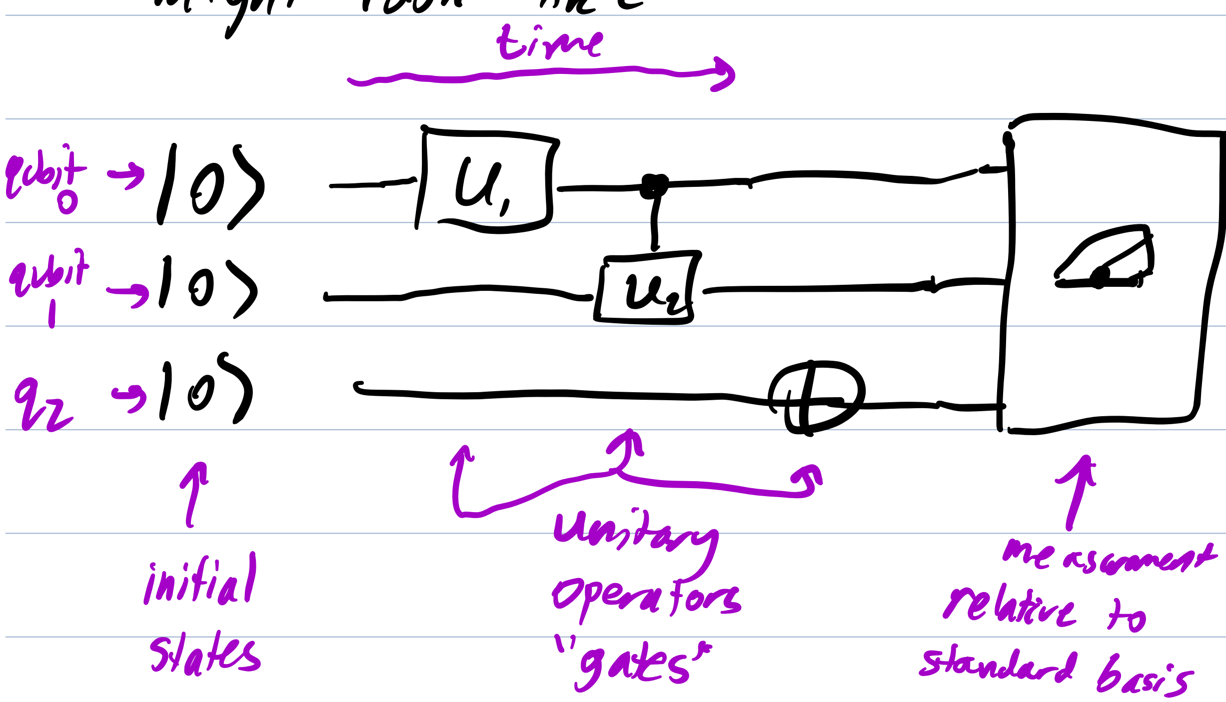
$\underbrace{00\ldots0}_{n}$ to $\underbrace{111\ldots1}_{n}$

we write $|k\rangle$ for $|e_k\rangle$

Example The 2-qubit quantum system has basis

$$|00\rangle, \ |01\rangle, \ |10\rangle, \ |11\rangle$$

A 3-qubit quantum circuit might look like



1-qubit circuits:

Quantum system = $\mathbb{C}^2$, with basis $\{|0\rangle, |1\rangle\}$

$$|0\rangle \;—\; \boxed{U_1} \;—\; \boxed{U_2} \;—$$

means initial state = $\mathbb{C}|0\rangle$

Step 1 : Apply the unitary U_1 to the curr. st.

Step 2 : Apply U_2 to the curr. st.

Common gates:

NOT gate $\oplus$

$|0\rangle \mapsto |1\rangle$

$|1\rangle \mapsto |0\rangle$

Hadamard gate $\boxed{H}$

$|0\rangle \mapsto \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

$|1\rangle \mapsto \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$